J. Fluid Mech. (1991), *vol.* 223, *pp.* 363–382
Printed in Great Britain

Lighthill quadrupole radiation in supersonic propeller acoustics

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(Received 1 November 1989 and in revised form 1 June 1990)

Sound generation by the Lighthill quadrupole is an important mechanism in the noise of supersonic and transonic propellers. Full numerical calculation of the quadrupole radiation must, however, require knowledge of the flow at all points exterior to the blades (involving transonic aerodynamics) and the evaluation of special functions. We describe how these difficulties may be largely avoided, using an asymptotic approximation that the number of blades, B, is large, and prove that to leading order the problem of the radiation in a given direction reduces to one of determining the (two-dimensional) flow field at just one radial station, legitimately achieved by linearized supersonic analysis. Simple formulae are derived for the far-field acoustic pressure generated by unswept blades, from which absolute level predictions can be made accurately and quickly. These formulae predict a significantly greater intensity, over broad angular ranges, than is predicted by the linear theory for thickness noise sources.

1. Introduction

The study of sound generation by rotating propeller blades has a long history, some of the earliest theoretical work having been carried out by Lynam & Webb (1919), and extended by such workers as Gutin (1936), Deming (1937, 1938) and Garrick & Watkins (1954).

However, the subject has taken on new importance over the last ten years, with the advent of the 'propfan', and related ultra-high-bypass engines, as possible powerplants for future generations of passenger aircraft. The wide variety of noise issues which have arisen, including the aeroacoustics of rapidly moving bodies, effects of complex blade geometry and the possible interactions between contra-rotating blade rows in certain propfan configurations, has attracted much research, both experimental and theoretical, up to the point where full-scale prototypes have been flight-tested on civilian transport aircraft. The key question, however, as yet unanswered, is whether such engines will satisfy existing noise regulations, and possibly more stringent ones in the future, relating to environmental noise, and whether they will lead to sufficiently low levels of cabin noise for passenger comfort.

It is therefore of great importance to develop accurate prediction schemes to handle these difficult problems, and much progress has been achieved, perhaps most notably by Hanson (1980, 1983), whose frequency-domain approach we adopt here, and by Farassat (1981). The basic formulation is in terms of the Ffowcs Williams & Hawkings (1969) equation, which is a generalization of the Lighthill acoustic analogy to include force and thickness terms. The equation gives an integral prescription for the sound field in terms of three distributed sources: monopoles, dipoles and quadrupoles. The monopoles and dipoles, corresponding to volume displacement effects and to lift and drag forces on the blade respectively, are distributed across the blade surface (and not in general the blade mid-chord), whilst the quadrupoles, associated with nonlinear effects at high speed, are spread over some volume around the blades. Calculation of the acoustic field requires, *a priori*, an explicit specification of the source terms, together with the motion of the surfaces, which will of course themselves depend on the global flow field. Even if the surface sources are supposed calculable by steady (and perhaps linear) aerodynamics, the problem of determining the quadrupole distribution remains. Here, indeed, explicit solutions have so far only been achieved for the most simple geometries and motions, e.g. a one-dimensional unsteady piston motion and a two-dimensional supersonic steady wedge motion (Blackburn 1982; Ffowcs Williams 1979).

Numerical studies have been taken further, for instance by Hanson & Fink (1979), who were able to calculate the quadrupole strength at each point on, and in the vicinity of, a rotating blade, and thereby estimate the additional noise levels due to the quadrupole nonlinearity. In their approach, however, the quadrupole input to the Ffowcs Williams & Hawkings equation must be calculated using steady aerodynamic codes, which proves especially difficult for those areas of the blade moving transonically, in addition to which certain generic Bessel functions must be computed at each point on the blade span and chord, at great expense in CPU time. Moreover, a full numerical calculation cannot possibly yield the same level of physical insight as a closed algebraic expression, nor provide any of the scaling laws so useful for design purposes.

In what follows here, we describe an approach which provides simple algebraic expressions for the quadrupole noise due to a supersonically rotating propeller, avoiding the necessity for transonic calculations and the evaluation of special functions. Our basic simplification, first suggested as a high-frequency approximation by Hawkings & Lowson (1974), and used to good effect by Parry & Crighton (1989*a*, *b*) and Crighton & Parry (1990*a*, *b*) in the evaluation of the monopole and dipole noise, is that the number of blades, B, is large. The frequency-domain radiation integrals can then be evaluated using standard asymptotic techniques, in the limit $mB \rightarrow \infty$, for all harmonic numbers $m = 1, 2, \ldots$. While this might seem an unwarranted simplification, previous experience has shown that even only $B = 4$ produces good agreement with a full numerical solution, and still better agreement for more typical modern values of 8 or 12. Following this procedure, only the quadrupole strength at the leading and trailing edges of just one blade cross-section need be calculated for any given observer location, which is achieved using just linearized supersonic flow theory, and represents a major simplification of the Hanson & Fink method.

A detailed analysis of the transonic flow region on the blade is in general not necessary, for the essence of the asymptotic technique is that it shows that, in the limit $B \rightarrow \infty$, the radiation in a given direction comes from a particular blade section, called the *Mach radius* (for that radiation angle), which is not a section of transonic flow, except for *reception* directions very close to the propeller disk plane. For such shallow reception angles, transonic flow codes are admittedly necessary, but even here the asymptotic technique helps considerably in limiting the information needed from those codes to the specification simply of the behaviour at the chordwise extremities of the blade at the Mach radius. Equally, in the case of a subsonic propeller, asymptotic theory would reduce the problem to one of determining the

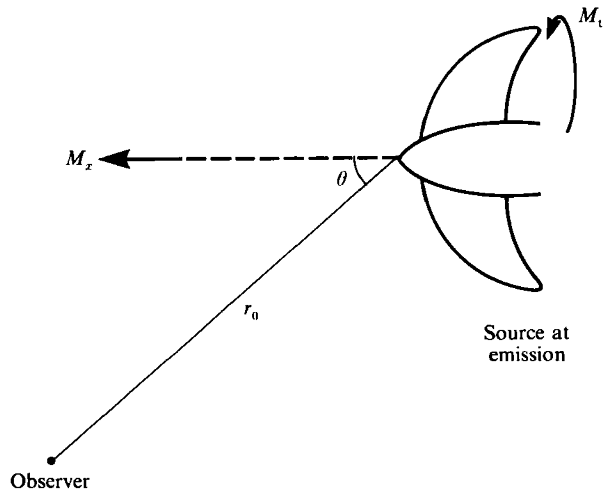

FIGURE 1. The definition of the emission coordinates, for an observer positioned in the horizontal plane, of a single-rotation propeller.

flow at just the blade tip, which would in itself be a very difficult problem numerically, but could well be amenable to experiment.

A rigorous application of the Ffowcs Williams & Hawkings equation requires that the surface terms (the steady loading and the thickness) be positioned on the genuine blade surfaces, and not, as is so often done, on the mean plane between them. The latter description is valid only when the source region is compact, and it has been shown (Ffowcs Williams & Hawkings 1969, §6), that transfer to the mean plane is certainly not adequate when the airfoil thickness is comparable with the Doppler-shifted wavelength, and in particular that great care must be exercised when considering Mach radiation, for which the effective Doppler frequency is infinite. Since Hanson's frequency-domain radiation integrals (taken as the starting point of our analysis) rely on the thin-blade approximation, and since the dominant contributions to the sound field at infinity will come from sources exactly satisfying the Mach wave condition, it is prudent to modify Hanson's (1980) work to include arbitrary blade thickness, and the new formula for the steady loading noise is given in Appendix C. It turns out that the error introduced in displacing the force singularities onto the midchord will be in the phase of the harmonics and in the higher derivatives of the steady loading real-time waveform, and that therefore use of the thin-blade approximation will be fully justified for calculating the lower harmonics of, and the sound pressure level due to, the surface dipole distribution. The essential reason is that azimuthal interference effects between the periodically spaced blades make compactness on the scale of the axially Doppler-contracted wavelength the sufficient condition for the thin-blade approximation. However, a qualitatively reliable estimate of the *total* radiated sound field (i.e. the sum of steady loading, thickness and quadrupole components) can only be made when proper account is taken of the non-vanishing airfoil thickness. Even for the lowest harmonics the error introduced by moving the force terms onto the midchord is of formally the same order in airfoil thickness (i.e. order thickness squared) as the Lighthill quadrupole term calculated in this paper, and, as argued above, this error becomes even more significant for large values of m. Accurate noise calculations for supersonic propellers therefore require the use of the Ffowcs Williams & Hawkings equation in its full form, and in the frequency domain this must entail use of the expression in Appendix C for the steady loading noise, together with the Lighthill radiation integral in equation (1).

It might prove possible to regard the calculation of the quadrupole fields as corresponding simply to the (nonlinear) propagation of the shocks attached to the blade edges – as indeed is suggested by the model calculations of Ffowcs Williams (1979) and Blackburn (1982) for one-dimensional unsteady or two-dimensional steady problems. However, the equivalence has not been proved in general, nor would it be helpful if one wants to calculate harmonic components of the sound in a frame fixed in the fluid at infinity. The propagation/distortion is equivalent to modification of the harmonic components, and the aeroacoustic analogy states that that modification can be calculated, as here, by evaluating the quadrupole integral.

2. Mathematical formulation

The starting point for our analysis is a modified form of an equation derived by Hanson (1980) for the mth harmonic of the acoustic pressure due to the Lighthill (1952) quadrupoles,

$$P_{\mathrm{q}}^{m} = -\frac{\rho_0 c_0^2 B \exp\left(\mathrm{i}mB\left[\frac{1}{2}\pi - \dfrac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right)}{4\pi\tilde{r}_0(1-M_x\cos\theta)}$$

$$\times\int_{z_0}^{1} M_{\mathrm{r}}^2 J_{mB}\left(\frac{mBzM_{\mathrm{t}}\sin\theta}{1-M_x\cos\theta}\right)[k_x^2\,\Psi_{11}+2k_x k_y\,\Psi_{12}+k_y^2\,\Psi_{22}-\tilde{c}^2\Phi]\,\mathrm{d}z. \qquad (1)$$

Here θ and $\tilde{r}_0$ are emission coordinates, measured by an observer in the far field; their meaning is made clear in figure 1. A tilde denotes normalization of a length with respect to blade span. The integration is along the radius (span) of the propeller, spanwise coordinate r being normalized with respect to the propeller radius to give z, which runs from z_0 at the hub to 1 at the tip; M_{r} is the helical Mach number at each radial station, M_{t} the tip rotational Mach number, and M_x the axial (flight) Mach number.

We now go on to consider the blade geometry. In Hanson's (1980) presentation of equation (1), helical coordinates γ_0 and ξ_0, along and perpendicular to the propeller–advance helix, are defined, and since the blade is supposed twisted (so that each section moves parallel to its chord), the system γ_0, ξ_0, r is non-orthogonal. Furthermore, derivation of (1) involves calculation of a covariant tensor, whose transformation properties are not the simple Cartesian ones assumed by Hanson, and therefore some modifications are needed to Hanson's analysis. In this paper we therefore use a Cartesian system (at rest in the fluid, but coincident with the blade at some early initial instant $t = 0$, well before the blade radiates) γ, ξ, r, where r is aligned along the radius of one blade at $t = 0$, and γ and ξ are perpendicular axes in the plane of the blade section at the Mach radius; see equation (8) ff. This is made clear in figure 2(a). Justification for this lies entirely in the large-B asymptotic formulation, and is in two stages. First, the z-integral of (1) is dominated (for mB large) by a region of length $(mB)^{-\frac{2}{3}}$ about the Mach radius, over which the angle of twist of Hanson's γ_0 and ξ_0 helicoidal coordinate axes will vary by an asymptotically small amount, permitting the use of just those axes defined at the Mach radius, at least to leading order. Second, the main contribution to the far-field quadrupole

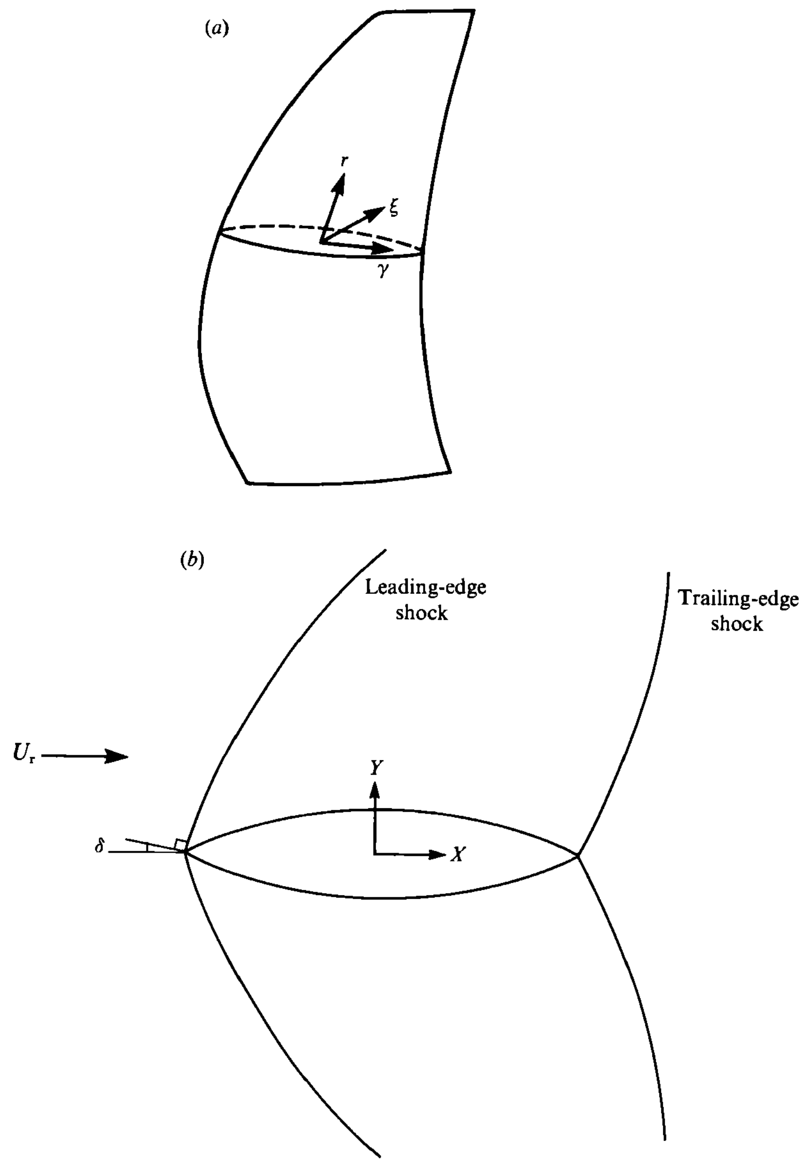

FIGURE 2. (*a*) The orthogonal coordinate system along a blade. (*b*) The airfoil cross-section with attached shocks at the leading and trailing edges.

radiation is localized close to the blade edges (equation (21) ff.), so that the curved helicoidal axes γ_0 and ξ_0 can be replaced by the Cartesian axes γ and ξ; the discrepancy between the two systems will again be asymptotically small, at least for small chord length. The whole process is entirely equivalent to the extension out to infinity of Cartesian axes localized at a stationary phase point. Therefore, Hanson's published quadrupole formulae can only be applied rigorously in the large-blade-number limit. A corrected derivation will be given in Peake & Crighton (1990), the results of which are exact for the non-orthogonal helicoidal coordinate system. The additional terms are found to give harmonic pressures smaller by a factor $O(mB)^{-1}$ than those quoted in this paper.

It is supposed that the blades are thin, so that thin-airfoil theory may be applied, and that the leading and trailing edges are sharp, so that any shocks remain attached. The normalized chord length $\tilde{c}$ will be supposed to scale on $B^{-\alpha_0}$, for some positive α_0; no attempt will be made in this paper to consider any cascade effects due to the change in propeller solidity with large B.

Wavenumbers k_x and k_y are defined by

$$\left.\begin{aligned} k_x &= \frac{mBM_{\mathrm{t}}\tilde{c}}{M_{\mathrm{r}}(1-M_x\cos\theta)}, \\ k_y &= \frac{mB\tilde{c}}{zM_{\mathrm{r}}}\left(\frac{M_{\mathrm{r}}^2\cos\theta-M_x}{1-M_x\cos\theta}\right), \end{aligned}\right\} \tag{2}$$

whilst Ψ_{ij} and Φ, functions of z, correspond to section-integrated quadrupole strengths, with account taken for retarded-time variations across the chord,

$$\left.\begin{aligned} \Psi_{ij} &= \iint T_{ij}\frac{\exp(-\mathrm{i}k_xX)\exp(-\mathrm{i}k_yY)}{\rho_0U_{\mathrm{r}}^2}\,\mathrm{d}X\,\mathrm{d}Y, \\ \Phi &= \iint \frac{\partial^2T_{33}}{\partial z^2}\frac{\exp(-\mathrm{i}k_xX)\exp(-\mathrm{i}k_yY)}{\rho_0U_{\mathrm{r}}^2}\,\mathrm{d}X\,\mathrm{d}Y, \end{aligned}\right\} \tag{3}$$

where U_{r} is the helical velocity of the blade cross-section, and the integration is taken over the whole plane of blade cross-section and X and Y correspond to γ and ξ, normalized by the blade chord length (figure 2*b*). Here T_{ij} is the Lighthill acoustic stress tensor, which is written to leading order in perturbation velocities in a form due to Schmitz & Yu (1979) as

$$T_{ij} = \rho_0u_iu_j+\tfrac{1}{2}(\Gamma-1)M_{\mathrm{r}}^2\rho_0u_1^2\delta_{ij}, \tag{4}$$

with Γ the ratio of specific heats. (Equation (4) is easily checked. One has exactly

$$T_{ij} = \rho_0u_iu_j+[(p-p_0)-c_0^2(\rho-\rho_0)]\,\delta_{ij}, \tag{5}$$

an assumed isentropic law
$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\Gamma} \tag{6}$$

and the standard result
$$\rho-\rho_0 = -\rho_0\left(\frac{U_{\mathrm{r}}^2}{c_0^2}\right)\frac{u_1}{U_{\mathrm{r}}} \tag{7}$$

of linearized supersonic aerodynamics.) The Φ term in (1) was missed out of Hanson's (1980) formulation; however, although it may be supposed that the radial velocity perturbation (i.e. u_3) is zero, there must still be a non-zero contribution to T_{33} from the isotropic part of the stress tensor (equation (4)). It is clear that, for large B, Ψ_{ij} and Φ are of the same order in mB, and hence that in (1) the last term is $O(mB)^{-2}$ smaller than the others, and will be neglected in the rest of this paper. For conventional propellers, with few blades, the T_{33} term would have to be included, and could become significant at high helical Mach numbers.

In the asymptotic limit of large blade number B, the dominant behaviour of (1) will be determined by the exponential dependence on mB and z in the integrand, which arises from the Bessel function and Ψ_{ij} factors. When k_x is held fixed (i.e. taken independent of z; see §5 for a modification to account for spanwise variation in k_x), Crighton & Parry (1990*a*) have demonstrated that the dominant behaviour of the sound field, in their case the steady loading and thickness components, is determined by the magnitude of

$$z^* = \frac{1-M_x\cos\theta}{M_t\sin\theta}, \tag{8}$$

referred to as the *Mach radius*. Mathematically, this is the point at which the Bessel function of (1) changes from exponential decay to rapid oscillation. For a propeller with subsonic tip speed, this parameter is always greater than unity, the acoustically weighted spanwise loading and thickness sources decay *exponentially* inboard from the tip $z = 1$, and so the largest contribution to P comes from the tip. However, for a supersonic propeller, z^* lies within the blade span for a significant range of observer positions, and so, by Laplace's method, it is precisely the source at the Mach radius that makes the main contribution to the pressure. Moreover, the expression of Schmitz & Yu for Ψ_{ij} (equation (4)) is only good at supersonic operating conditions, and so in this paper we do not address the question of quadrupole radiation generated by subsonic blade motion.

The defining property of the Mach radius is that its velocity component at emission, in the direction of the observer, is exactly sonic at some point in each revolution, i.e.

$$M_t z^*\sin\theta+M_x\cos\theta = 1. \tag{9}$$

However, the absolute velocity of such a point will be supersonic; for instance, the helical Mach number of the station z^* corresponding to $\theta = \frac{1}{2}\pi$ would be $(1+M_x^2)^{\frac{1}{2}}$, well into the supersonic regime at cruise conditions, while even at the low values of M_x relevant to take-off operations the helical Mach number at $z^*(\theta)$ will be supersonic for many angles ahead of the propeller, $0<\theta<\frac{1}{2}\pi$.

The expression in (1) can be characterized by the form

$$P_q^m = \int_{z_0}^{1} S(z)\, J_{mB}\left(\frac{mBz}{z^*}\right)\mathrm{d}z. \tag{10}$$

It has been demonstrated (Hanson & Fink 1979) that Ψ_{ij}, and hence S, is maximal for those blade sections travelling in the transonic regime (in the neighbourhood of the *sonic radius* $z = z_s$), and is small elsewhere; according to linearized supersonic theory, the quadrupole strength will vary as $(M_r^2-1)^{-1}$, inversely with distance from the sonic radius. However, as noted above, the Bessel function J_{mB} decays exponentially rapidly inboard of the Mach radius, and certainly very much faster than the decay of $S(z)$ away from $z = z_s$. Since it is always the case that $z_s \leqslant z^*$ (with equality only at one critical observer position defined later), the contribution from the sonic radius must be negligibly (in fact, exponentially) small compared with that from the Mach radius, a fact not apparent when a purely numerical evaluation of (1) is performed.

Therefore, to obtain the first term in an asymptotic expansion in large B it is only necessary to calculate the source strength at the Mach radius, P_q^m becoming

$$S(z^*)\frac{z^*}{|mB|}+o\left(\frac{1}{|mB|}\right). \tag{11}$$

This is derived rigorously in Appendix A. In marked contrast, a numerical solution would need to calculate the quadrupole over a whole range of radial stations, not merely at a single station. In the next section we demonstrate how an algebraic expression for $S(z^*)$ can be found simply from two-dimensional theory.

3. Calculation of the quadrupole

We adopt the same approach as Hanson & Fink (1979), who calculate the quadrupole strength associated with two-dimensional airfoil sections along the blade using steady subsonic, supersonic and transonic aerodynamic theory, depending on the local helical Mach number. The great simplification of the asymptotic approach is that only the flow field at the Mach radius need be calculated, and, since the local motion there is generally well into the supersonic regime, the necessity of complex transonic calculations is avoided. Additionally, it will be seen that Ψ_{ij} is dominated by contributions from small regions very close to the leading and trailing edges, so the assumption of two-dimensionality will be a good one, provided of course that the Mach radius is not too close to the tip. The analysis below assumes that each airfoil section is in uniform rectilinear motion; the linear solution for a rotating blade is rather different, and has been described by Chapman (1988), and specifically, the Mach surface consists of a 2-sheeted cusped cone. Near the back face

$$p \sim \tilde{p} \ln |\boldsymbol{x}-\boldsymbol{x}_{\mathrm{s}}(t)|,$$

and the corresponding T_{ij} is therefore singular like $\ln^2|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{s}}|$. The contribution to Ψ_{ij} is, by precisely the arguments leading to (20) and (21) below, comparable with the leading-edge shock contribution, and not surprisingly, because $\ln^2|x|$ and $H(x)$ have essentially the same asymptotic Fourier transform. However, nonlinear effects excluded from Chapman's analysis must imply that the rear sheet singularity will lead to contributions which are smaller, for large mB, than those associated with the leading-edge shocks, which must survive nonlinear modification. Similar remarks apply to the contribution from the cusped line, where the pressure on linear theory has a weak algebraic singularity. In any event we deal here with only the leading- and trailing-edge shocks. The relevance of the singularities of linear propeller theory to the practical problem remains an open question and will probably need numerical solution of the full nonlinear equations.

Accordingly, the problem reduces to calculating the two-dimensional flow over a thin airfoil, with a uniform supersonic upstream velocity $U_{\mathrm{r}} = M_{\mathrm{r}} c_0$, parallel to the airfoil chord. We suppose that the flow is inviscid and isentropic, and assume small perturbation velocities u_1 and u_2, parallel and perpendicular to the chord. The blade is taken to be symmetric about its chord, and has surface $Y = h(X)$, leading and trailing edges being sharp. Under these approximations, the density ρ is

$$\rho = \rho_0\left(1 - M_{\mathrm{r}}^2 \frac{u_1}{U_{\mathrm{r}}} + \ldots\right). \tag{12}$$

A strictly linear solution predicts that there will be shocks attached to the leading and trailing edges, parallel to the Mach lines, and with constant strength right out to infinity; the integral defining Ψ_{ij} would not be defined for such a flow. However, owing to accumulating nonlinearities, this solution breaks down far from the airfoil, although the linear theory is still valid for points on and near the blade surface. Calculation of Ψ_{ij} therefore requires a uniformly valid expansion for $\boldsymbol{u}$, such as that given by Van Dyke (1964) or Caughey (1969).

Van Dyke (1964) defines a strained coordinate s, implicitly given by

$$\left.\begin{aligned}\gamma-\beta\xi &= s-\tfrac{1}{2}(\Gamma+1)\frac{M_{\mathrm{r}}^4}{\beta^2}Yh'(s)+\ldots,\\ \beta &= (M_{\mathrm{r}}^2-1)^{\frac{1}{2}},\end{aligned}\right\} \tag{13}$$

which Caughey (1969) determines explicitly for a circular-arc airfoil. Then a uniformly valid first-order approximation to the perturbation velocity above the airfoil is

$$\left.\begin{aligned}u_1 &= -\frac{U_{\mathrm{r}}h'(s)}{\beta}+\ldots,\\ u_2 &= U_{\mathrm{r}}h'(s)\ldots,\end{aligned}\right\} \tag{14}$$

between shocks originating at the leading and trailing edges $X=\pm\frac{1}{2}$, and with $\boldsymbol{u}$ zero elsewhere. Of course, u_1 will be symmetric about $Y=0$, and u_2 antisymmetric. These shocks are not the straight lines predicted by linear theory, but are curved, and decay in strength with distance from the airfoil. For our purposes, it is only necessary to note that u_1, u_2 become zero at infinity; an explicit inversion of (13) is not required.

It now follows that

$$\Psi_{ij}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{u_i u_j}{U_{\mathrm{r}}^2}+\tfrac{1}{2}(\Gamma-1)M_{\mathrm{r}}^2\frac{u_1^2}{U_{\mathrm{r}}^2}\delta_{ij}\right\}\exp(-\mathrm{i}k_x X)\exp(-\mathrm{i}k_y Y)\,\mathrm{d}X\,\mathrm{d}Y, \tag{15}$$

where $u_i u_j$ can be written as

$$u_i u_j = \alpha_{ij}[h'(s)]^2U_{\mathrm{r}}^2\{H(X-x_{\mathrm{L}})-H(X-x_{\mathrm{T}})\}, \tag{16}$$

with

$$\alpha_{ij}=\begin{pmatrix}\dfrac{1}{\beta^2} & -\dfrac{\operatorname{sgn}(Y)}{\beta}\\ -\dfrac{\operatorname{sgn}(Y)}{\beta} & 1\end{pmatrix}, \tag{17}$$

x_{L}, x_{T} being the X-coordinates of the leading and trailing shocks, as functions of Y, and H denoting the unit step function.

Substitution into (15), followed by use of Lighthill's (1958) asymptotic Fourier transform method yields, for large mB,

$$\begin{aligned}\Psi_{ij} \sim {}& \frac{\mathrm{i}}{k_x}\int_0^{\infty}\Big[[h'(s)]^2\{\alpha_{ij}+\tfrac{1}{2}(\Gamma-1)M_{\mathrm{r}}^2\alpha_{11}\delta_{ij}]\exp(-\mathrm{i}k_x X)\Big]_{x_{\mathrm{L}}}^{x_{\mathrm{T}}}\exp(-\mathrm{i}k_y Y)\,\mathrm{d}Y\\ &+\frac{\mathrm{i}}{k_x}\int_{-\infty}^{0}\Big[[h'(s)]^2\{\alpha_{ij}+\tfrac{1}{2}(\Gamma-1)M_{\mathrm{r}}^2\alpha_{11}\delta_{ij}\}\exp(-\mathrm{i}k_x X)\Big]_{x_{\mathrm{L}}}^{x_{\mathrm{T}}}\exp(-\mathrm{i}k_y Y)\,\mathrm{d}Y,\end{aligned} \tag{18}$$

and we refer to the two terms in (18) as Ψ_{ij}^{+} and Ψ_{ij}^{-} respectively. We note that to leading order in airfoil thickness X_Y, which is just the X-coordinate of the leading or trailing shocks differentiated with respect to Y, takes the value β at both the leading and trailing edges for $Y = 0^+$, and the value $-\beta$ for $Y = 0^-$. The asymptotic evaluation of (18) will now depend on the value of the quantity $k_y + k_x X_Y$ at the leading and trailing edges, just above and below the airfoil (i.e. at $Y = 0^+$ and $Y = 0^-$), so we write

$$k_y + k_x X_Y = O(mB)^\alpha \tag{19}$$

(where appropriate values of α will be discussed in a moment), and note that if for instance $\alpha > 0$ in Ψ^+_{ij}, a further integration by parts furnishes

$$\Psi^+_{ij} \sim \frac{1}{k_x}\left[\frac{1}{k_y + k_x X_Y}\{\alpha_{ij} + \tfrac{1}{2}(\Gamma - 1) M_r^2 \alpha_{11}\delta_{ij}\}[h'(s)]^2 \exp(-\mathrm{i}k_x X)\right]_{-\frac{1}{2}, 0^+}^{\frac{1}{2}, 0^+}, \tag{20}$$

indicating that the leading-order contributions to the integrated quadrupole strength come precisely from the leading and trailing edges.

On the other hand, for $\alpha < 0$ we can expand the phase function up to quadratic terms, and hence there will effectively be a stationary point of the integrand at $Y = 0$, and hence the method of stationary phase yields that

$$\Psi^+_{ij} \sim \frac{\mathrm{i}}{2k_x}\left[\left(\frac{2\pi}{|X_{YY}| k_x}\right)^{\frac{1}{2}}\{\alpha_{ij} + \tfrac{1}{2}(\Gamma - 1) M_r^2 \alpha_{11}\delta_{ij}\}[h'(s)]^2 \right.$$
$$\left. \times \exp(-\mathrm{i}k_x X + \tfrac{1}{4}\mathrm{i}\pi \,\mathrm{sgn}\,[X_{YY}])\right]_{-\frac{1}{2}, 0^+}^{\frac{1}{2}, 0^+}, \tag{21}$$

again demonstrating the leading- and trailing-edge dominance. The formulae equivalent to (20) and (21) for Ψ^-_{ij} follow in exactly the same way. The quantities X_Y and X_{YY} can be calculated for individual airfoil shapes, but for the sake of definiteness we shall consider the symmetric circular-arc airfoil, given by

$$h(X) = \frac{b}{c}(1 - 4X^2), \tag{22}$$

where b is the maximum blade thickness, and the thickness : chord ratio b/c is small. It should be emphasized that the analysis could be repeated for any given airfoil with a sharp nose. Caughey (1969) derives the following expressions for the shock gradients in the flow over a circular-arc airfoil, correct to first order in thickness:

$$X_Y|_{Y=0^+} = \beta \mp \frac{(\Gamma+1) M_r^4}{2\beta^2}\frac{b}{c} + \ldots, \tag{23}$$

and

$$X_{YY}|_{Y=0^+} = \pm \frac{(\Gamma+1)^2 M_r^8}{2\beta^4}\left(\frac{b}{c}\right)^2 + \ldots, \tag{24}$$

where the upper and lower signs refer to leading and trailing edges respectively. It is easy to see, just from the symmetric location of the shocks about $Y = 0$, that $X_Y|_{Y=0^-} = -X_Y|_{Y=0^+}$ and $X_{YY}|_{Y=0^-} = X_{YY}|_{Y=0^+}$.

We shall now consider what value α must take under typical operating conditions, and for various observer positions. It can be shown by simple manipulation that

$$k_y \pm k_x \beta = \frac{mB\tilde{c}M_t}{M_r(1 - M_x \cos\theta)}\left[\frac{\cos\theta - M_x}{\sin\theta} \pm \frac{|M_x - \cos\theta|}{\sin\theta}\right], \tag{25}$$

where all quantities are evaluated at the Mach radius, and the modulus arises since β is always positive. There is a critical observer position, given by $\cos\theta = M_x$ (corresponding to a reception angle of $\frac{1}{2}\pi$), where both β and (25) vanish, and hence the Mach radius for this θ coincides exactly with the sonic radius. Moreover, the α of (19) would be zero, and an asymptotic evaluation no longer possible. In the vicinity of this critical point our linearized supersonic theory will break down, and transonic analysis becomes necessary. However, our theory is perfectly valid on both sides of a narrow region around this value of θ, provided of course that the Mach radius remains within the blade span. For $\cos\theta$ strictly greater than M_x, it can be seen from (25) that $k_y + k_x\beta$ is large, and thus from (23) and (19) that $\alpha > 0$, and that the expression for Ψ_{ij}^{+} in (20) holds. Conversely, for $\cos\theta$ strictly less than M_x, then $k_y + k_x\beta$ will vanish identically, and (23) gives

$$k_y + k_x X_Y = \mp \frac{(\Gamma+1) M_{\mathrm{r}}^3 \tilde{b} m B M_{\mathrm{t}}}{2\beta^2(1-M_x\cos\theta)}. \tag{26}$$

Therefore, for sufficiently thin blades, $\alpha < 0$ and Ψ_{ij}^{+} is represented by (21). For thick blades, or for the higher harmonics, or for a very large number of blades (for instance in fans), the right-hand side of (26) would be large, and $\alpha > 0$ here as well, but in the rest of this paper we shall restrict ourselves to consideration of typical propfan parameters, and take the α of Ψ_{ij}^{+} to be negative when $\cos\theta < M_x$. Now considering the Ψ_{ij}^{-} term, we have for $\cos\theta$ strictly greater than M_x that $k_y - k_x\beta$ is zero (and hence $\alpha < 0$), whilst for $\cos\theta$ strictly less than M_x, it follows that $k_x - k_y\beta$ is large, and $\alpha > 0$.

To summarize then, when $\cos\theta > M_x$, the term Ψ_{ij}^{+} is given by (20), and Ψ_{ij}^{-} by the equation equivalent to (21), whilst for $\cos\theta < M_x$ the situation is reversed.

The expression $k_y + k_x X_Y = 0$ corresponds to the leading (or trailing) shocks having an exactly sonic velocity component in the observer direction (proved in Appendix B); one would expect the shocks to satisfy this sonic condition exactly, and not merely to leading order in thickness (cf. (26)). The explanation for this is that Hanson's formula, (1), has been developed by applying boundary conditions at the airfoil midchord, thus neglecting any thickness (at least in the wavenumbers k_x and k_y), and therefore (under this approximation) the Ffowcs Williams & Hawkings sonic condition (for the shocks above the airfoil) reduces to $k_y + k_x\beta = 0$, satisfied for observers positioned rearward of the critical observer position. For the shocks below the airfoil, the sonic condition becomes $k_y - k_x\beta = 0$, which is in turn satisfied for observers ahead of the critical region. The effect of non-zero thickness is to change the angle of inclination of the shocks to the X-axis (cf. (23)), but such shocks would presumably still satisfy the sonic condition exactly, if proper account could be taken of blade thickness.

It can easily be seen that the value of Ψ_{ij}^{+} in (20) (i.e. when the upper shock does not satisfy the sonic condition) is smaller, by a factor $(mB)^{-\frac{1}{2}}$, than the value given by (21) (i.e. when the sonic condition *is* satisfied by the upper shock). It therefore follows that behind the critical observer position it is the upper shock that contributes most strongly to the radiation, whilst ahead of $\cos\theta = M_x$ it is the lower shock that dominates. In comparison, the remaining contributions, as defined in (20), are small and can be neglected.

It is worth remarking that the integral in (18) will converge because of the decay in perturbation velocity away from the airfoil. For instance, for a circular-arc airfoil, it is well known that the perturbation velocity falls off like $Y^{-\frac{1}{4}}$, guaranteeing convergence. Such a decay is rather slow, and is of itself not sufficient to validate our taking two-dimensional sections along the blade. However, it is seen in (20) and (21) that it is the large value of mB that ensures that the main contributions to Ψ_{ij} come from the leading and trailing edges, and that the original assumption of two-dimensionality is a good one.

The leading edge/trailing edge dominance of this sound field is hardly surprising, given the non-smoothness of the blade slope at these points. This is also apparent in the similar problem of a thin airfoil accelerated in a straight line through supersonic velocities (see Lilley *et al.* 1953), where the characteristic 'arrow-head' shock structure (being the envelope of wavelets emitted by the leading and trailing edges) propagates out to the far field, so that the waveform received at infinity is effectively that generated by point sources located at the sharp edges.

4. Slow variation in chordwise wavenumber

Since the chord length c will be rather smaller than the blade diameter D, a first approximation that k_x is constant along the blade span will be made; certainly the oscillation of the exponential factor in (20) and (21) will be much less rapid than that of the Bessel function of (1). In that case, we suppose that k_x takes its value at the Mach radius, so that, as remarked in §2, the dominant contribution to P will come from the Mach radius (8). At this point the integrated quadrupole strength, for an observer at $\cos\theta > M_x$, is given by an equation of the form (21), as

$$\Psi_{ij} \sim \left(\frac{2\pi}{|X_{YY}|}\right)^{\frac{1}{2}} [h'(\tfrac{1}{2})]^2 \left[\alpha_{ij} + \tfrac{1}{2}(\Gamma-1)M_{\mathrm{r}}^2 \alpha_{11}\,\delta_{ij}\right] \frac{\sin(\frac{1}{2}k_x + \frac{1}{4}\pi)}{k_x^{\frac{3}{2}}}, \tag{27}$$

on neglect of the contribution from Ψ_{ij}^{+}. Then substitution of (27) into (11) yields a final expression

$$P_{\mathrm{q}}^m \sim -\frac{8\rho_0 c_0^2 M_{\mathrm{r}}^{-2}\beta^2}{\pi^{\frac{1}{2}} m \sin\theta M_{\mathrm{t}}\tilde{r}_0(\Gamma+1)} \exp\left(imB\left[\tfrac{1}{2}\pi - \frac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right) \times\left(\frac{b}{c}\right)\frac{\sin(\frac{1}{2}k_x+\frac{1}{4}\pi)}{k_x^{\frac{3}{2}}}\left\{\left(\frac{k_x}{\beta}+k_y\right)^2 + \tfrac{1}{2}(\Gamma-1)\frac{M_{\mathrm{r}}^2}{\beta^2}(k_x^2+k_y^2)\right\} \tag{28}$$

for the quadrupole pressure, forward of the critical observer position, the analysis having been performed to leading order in thickness. Similarly, using (21), the pressure for observer angles strictly greater than $\cos^{-1}(M_x)$ is given by

$$P_{\mathrm{q}}^m \sim -\frac{8\rho_0 c_0^2 M_{\mathrm{r}}^{-2}\beta^2}{\pi^{\frac{1}{2}} m \sin\theta M_{\mathrm{t}}\tilde{r}_0(\Gamma+1)} \exp\left(imB\left[\tfrac{1}{2}\pi - \frac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right) \times\left(\frac{b}{c}\right)\frac{\sin(\frac{1}{2}k_x+\frac{1}{4}\pi)}{k_x^{\frac{3}{2}}}\left\{\left(\frac{k_x}{\beta}-k_y\right)^2 + \tfrac{1}{2}(\Gamma-1)\frac{M_{\mathrm{r}}^2}{\beta^2}(k_x^2+k_y^2)\right\}, \tag{29}$$

on neglect of the contribution from the lower shock. Whilst all this analysis has been performed for $m > 0$, the harmonic components for negative m can be simply obtained from the above by complex conjugation. In (28) and (29), k_x, k_y, β and M_{r} are all to be evaluated at the Mach radius $z = z^*$.

These expressions for P_{q}^m are clearly very much simpler than (1), while retaining full dependence on all the relevant parameters. The predicted sound pressure level

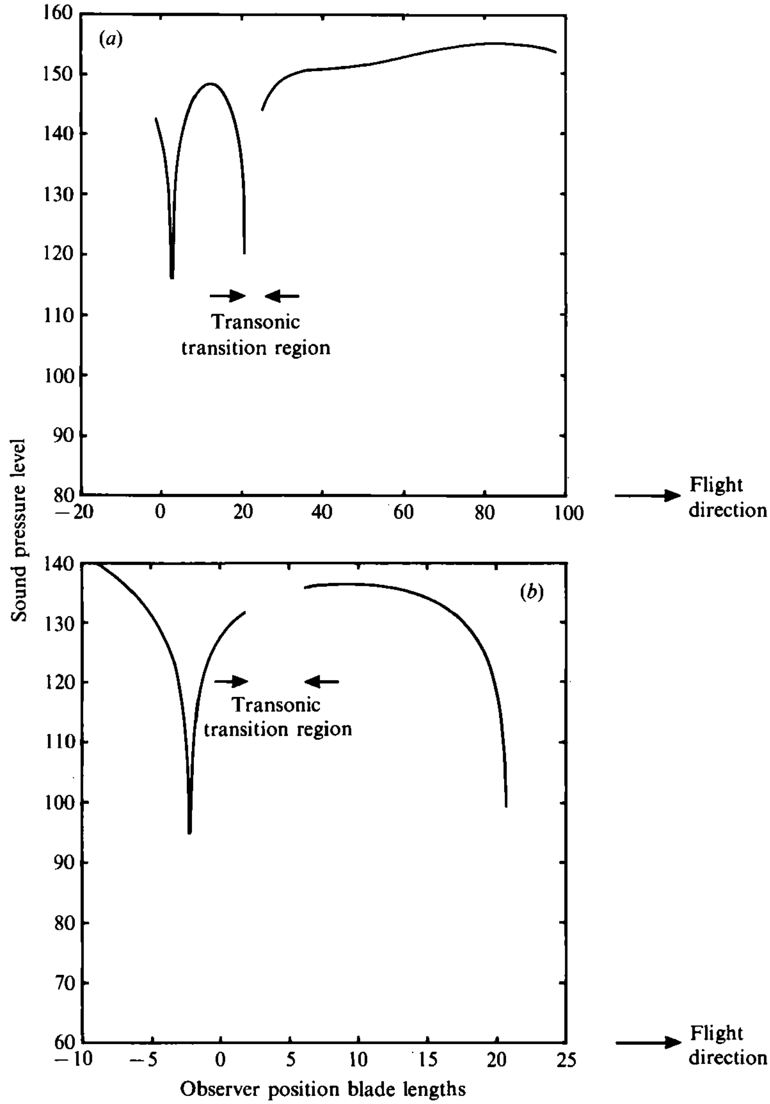

FIGURE 3. A prediction of the absolute sound pressure level (for the first harmonic) due to the quadrupoles of a 12-bladed single-rotation propeller, with $M_t = 1.4$, $\tilde{c} = 0.3$ and $b/c = 0.02$. The observer is positioned at a distance of 20 blade lengths from the propeller axis, and makes a traverse parallel to the axis; the abscissa is the observer position, relative to the propeller at emission. (*a*) The flight Mach number is taken as $M_x = 0.75$, typical of cruise conditions. (*b*) The flight Mach number is taken as $M_x = 0.2$, typical of take-off conditions.

due to the quadrupoles has been calculated, and is plotted in figure 3(*a*) for typical parameters of supersonic tip rotation at cruise, and in figure 3(*b*) at take-off. The range of these plots has been chosen so that the Mach radius remains within the blade span for all observer positions considered. This is achieved simply by setting $z^* = 1$ in (8), which yields the bounding values of θ as given by

$$\cos\theta_{\pm} = \frac{M_x \pm M_t(M_x^2+M_t^2-1)^{\frac{1}{2}}}{M_x^2+M_t^2}, \tag{30}$$

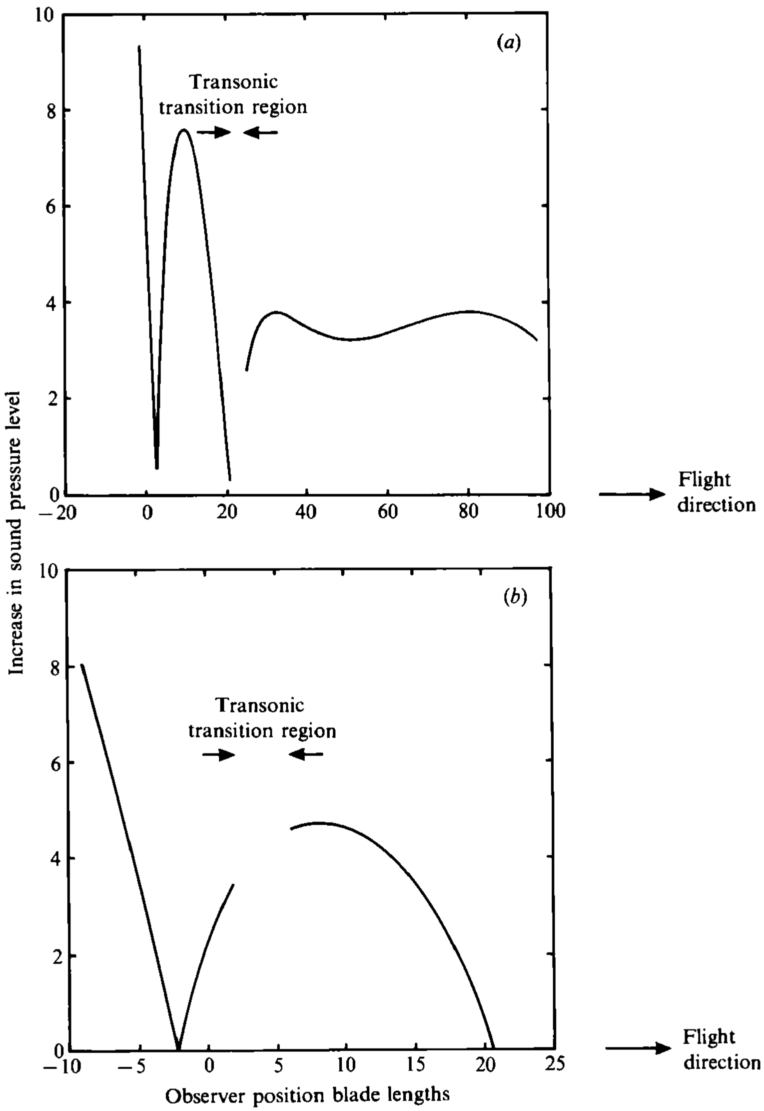

FIGURE 4. A prediction of the increase in sound pressure levels due to the quadrupole radiation, above those predicted by linear theory: (a, b) conditions as in figure 3(a, b) respectively.

so that our analysis is valid for observer positions within $\cos^{-1}\theta_+ < \theta < \cos^{-1}\theta_-$. Outside this angular range, the noise will be tip dominated, and analytical calculation of the required quadrupole would no longer be possible. In figure 4(a, b) estimates are made of the effects of quadrupole radiation over and above linear theory at cruise and take-off; under these operating conditions the dominant surface source will be the thickness term, the noise due to which is calculated simply, using Crighton & Parry's (1990a) asymptotic approximation. The approximation is (Crighton & Parry 1990a, equations (4) and (9))

$$P_{\mathrm{v}}^{m} \sim -\frac{\rho_0 c_0^2 M_{\mathrm{r}}^2}{4\pi m M_{\mathrm{t}} \tilde{r}_0 \sin\theta} \frac{b}{c} k_x^2 \exp\left(\mathrm{i}mB\left[\tfrac{1}{2}\pi - \frac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right), \tag{31}$$

where the notation is that of the present paper. The quantity plotted in figure 4(a, b) is $20 \log_{10} P_{\mathrm{q}}^{m} / P_{\mathrm{v}}^{m}$. The quadrupole is seen to be significant over the whole range of observer positions considered.

5. Modification to include variation of wavenumber near the Mach radius

In their (1990) paper, Parry & Crighton describe a method for including the effects of blade sweep in the asymptotic scheme for a supersonic propeller, by linearizing about the Mach radius. Exactly the same procedure may be used to account for phase variations along the blade span.

Formally, the problem is to evaluate the first term in the asymptotic expansion of

$$I = \int_{z_0}^{1} P(z)\, J_{mB}\left(mB \frac{z}{z^*}\right) \exp\left(-\tfrac{1}{2} \mathrm{i} k_x\right) \mathrm{d}z, \tag{32}$$

where $P(z)$ contains no exponential dependence on mB and z. The dominant contribution will come from the vicinity of the Mach radius, so that

$$I \sim \int_{z_-}^{z_+} P(z)\, J_{mB}\left(mB \frac{z}{z^*}\right) \exp\left(-\tfrac{1}{2} \mathrm{i} k_x\right) \mathrm{d}z, \tag{33}$$

where the limits of integration

$$z_{\pm} = z^*\{1 \pm \tfrac{1}{2}[(mB)^{-\frac{1}{3}}]\} \tag{34}$$

can be determined by standard arguments. We linearize the problem about z^*, so that

$$\left.\begin{aligned} \tfrac{1}{2} k_x &= \tfrac{1}{2} \bar{k}_x + \Delta \left(\frac{z}{z^*} - 1\right) mB, \\ \Delta &= -\frac{2 M_{\mathrm{t}}^3 z^{*2} c/D}{(1 - M_x \cos\theta) M_{\mathrm{r}}^3}, \end{aligned}\right\} \tag{35}$$

with c, M_{r}, $\bar{k}_x$ evaluated at $z = z^*$. Now making the transformation

$$z' = (mB)^{\frac{1}{3}} \left(\frac{z}{z^*} - 1\right) \tag{36}$$

and using the asymptotic expansion of J_{mB} in terms of an Airy function, and an integral representation of that Airy function, Parry & Crighton show that

$$I \sim \frac{z^* P(z^*) \exp\left(-\tfrac{1}{2} \mathrm{i} \bar{k}_x\right)}{2\pi mB} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left[\frac{\mathrm{i} t^3}{6mB} - \mathrm{i} z' t - \mathrm{i} \Delta (mB)^{\frac{2}{3}} z'\right] \mathrm{d}t\, \mathrm{d}z'. \tag{37}$$

By making the substitution $t = (mB)^{\frac{2}{3}} T$, interchanging the order of integration, and performing the z' integration, one finds

$$I \sim \sum_{\pm} \pm \frac{z^* P(z^*)}{mB} \exp\left[-\tfrac{1}{2} \mathrm{i} \bar{k}_x \pm \tfrac{1}{2} \mathrm{i} (mB)^{\frac{2}{3}} \Delta\right] \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\exp\left[\tfrac{1}{2} \mathrm{i} (mB)^{\frac{1}{2}} \left(\tfrac{1}{3} T^3 \pm T\right)\right]}{\left[T + (mB)^{\frac{1}{6}} \Delta\right]} \mathrm{d}T. \tag{38}$$

There are two dominant components to this integral, the first from the stationary phase points of the exponential, and the second from the pole in the integrand at

$$T = -(mB)^{\frac{1}{6}} \Delta. \tag{39}$$

The former can be calculated by standard analysis, but to find the pole contribution we make the transformations $T+(mB)^{\frac{1}{6}}\Delta \to u$, followed by $u \to w$, where w is defined by

$$w((mB)^{\frac{1}{3}} \pm 1) = \tfrac{1}{3}u^3 - (mB)^{\frac{1}{6}}\Delta u^2 + [(mB)^{\frac{1}{3}}\Delta^2 \pm 1]\,u, \tag{40}$$

and it follows that the pole contribution to each term is

$$I_{\pm}^{\text{pole}} = \pm\frac{z^*P(z^*)}{mB}\exp\left[-\tfrac{1}{2}\mathrm{i}\bar{k}_x - \tfrac{1}{6}\mathrm{i}mB\Delta^3\right]\tfrac{1}{2}\operatorname{sgn}\left[(mB)^{\frac{1}{3}}\Delta^2 \pm 1\right]. \tag{41}$$

When $|(mB)^{\frac{1}{6}}\Delta| > 1$, the contribution from the pole is zero and so the leading term in I comes from the stationary phase points, i.e.

$$I \sim \frac{z^*P(z^*)}{mB}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(mB)^{-\frac{1}{4}}\sin\{\tfrac{1}{3}[(mB)^{\frac{1}{2}}] - \tfrac{1}{4}\pi\}\exp\left[-\tfrac{1}{2}\mathrm{i}\bar{k}_x - \tfrac{1}{2}\mathrm{i}(mB)^{\frac{2}{3}}\Delta\right]. \tag{42}$$

Conversely, when $|(mB)^{\frac{1}{6}}\Delta| < 1$ the pole makes a non-zero contribution, which dominates that of the stationary points, so that

$$I \sim \frac{z^*P(z^*)}{mB}\exp\left[-\tfrac{1}{2}\mathrm{i}\bar{k}_x - \tfrac{1}{6}\mathrm{i}mB\Delta^3\right]. \tag{43}$$

Transition through the condition $|(mB)^{\frac{1}{6}}\Delta| = 1$ needs a separate analysis, which will not be given here.

These results are now applied to our quadrupole problem, substituting (20) or (21) into (1), and then using (42) and (43), together with their complex conjugates, to evaluate the subsequent integrals asymptotically. Thus we arrive at two quite distinct cases for the quadrupole noise; we just present the $\cos\theta > M_x$ case here.

Case A $|(mB)^{\frac{1}{6}}\Delta| < 1$

$$P_{\mathrm{q}}^m \sim -\frac{8\rho_0 c_0^2 M_{\mathrm{r}}^{-2}\beta^2}{\pi^{\frac{1}{2}} m\sin\theta M_{\mathrm{t}}\tilde{r}_0(\Gamma+1)}\exp\left(\mathrm{i}mB\left[\tfrac{1}{2}\pi - \frac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right)$$
$$\times\frac{b/c}{k_x^{\frac{3}{2}}}\sin\left(\tfrac{1}{2}k_x + \tfrac{1}{4}\pi + \tfrac{1}{6}mB\Delta^3\right)\left\{\left(\frac{k_x}{\beta}+k_y\right)^2 + \tfrac{1}{2}(\Gamma-1)\frac{M_{\mathrm{r}}^2}{\beta^2}(k_x^2+k_y^2)\right\}, \tag{44}$$

where the bars on k_x have been dropped. Note that the case $\Delta = 0$ reduces to equation (28).

Case B $|(mB)^{\frac{1}{6}}\Delta| > 1$

$$P_{\mathrm{q}}^m \sim -\frac{2^{\frac{7}{2}}\rho_0 c_0^2 M_{\mathrm{r}}^{-2}\beta^2}{\pi m\sin\theta M_{\mathrm{t}}\tilde{r}_0}(\Gamma+1)(mB)^{-\frac{1}{4}}\exp\left(\mathrm{i}mB\left[\tfrac{1}{2}\pi - \frac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right)$$
$$\times\frac{b/c}{k_x^{\frac{3}{2}}}\sin\left[\tfrac{1}{2}k_x + \tfrac{1}{4}\pi + (mB)^{\frac{2}{3}}\tfrac{1}{2}\Delta\right]\sin\{\tfrac{1}{3}[(mB)^{\frac{1}{2}}] - \tfrac{1}{4}\pi\}$$
$$\times\left\{\left(\frac{k_x}{\beta}+k_y\right)^2 + \tfrac{1}{2}(\Gamma-1)\frac{M_{\mathrm{r}}^2}{\beta^2}(k_x^2+k_y^2)\right\}. \tag{45}$$

Case A corresponds to a relatively slow variation of k_x along the blade span, the effect of which is merely to introduce a phase shift of $\frac{1}{6}mB\Delta^3$ into (28). On the other hand, in Case B the wavenumber varies rapidly round the Mach radius, introducing a more drastic modification to (28), which we shall not consider further, since for all cases of practical interest Δ is small.

6. Conclusions

In this paper explicit expressions for the far-field acoustic pressure generated by the Lighthill quadrupoles of a supersonic propeller have been given, using the asymptotic approximation $B \to \infty$. The noise has been proved to be dominated by a particular radial station, the Mach radius, the flow around which is simply determined by linearized supersonic theory. In the asymptotic limit, it is shown that only the leading and trailing edges of the Mach-radius blade cross-section contribute to leading order, in justification of previous assumptions of two-dimensionality. In the vicinity of a critical observer position $\cos\theta = M_x$, the flow is transonic, but away from this point simple analytical formulae have been found.

An absolute prediction of the pressure level due to the quadrupoles can now be easily performed, avoiding most of the problems associated with transonic flow and calculation of special functions encountered in other work on the subject.

The authors are particularly grateful to J. E. Ffowcs Williams and C. J. Chapman for many illuminating discussions. N.P. acknowledges receipt of an SERC studentship and the support of Rolls-Royce plc.

Appendix A

We need to evaluate the first term of the asymptotic expansion for large mB of the integral

$$I = \int_{z_0}^{1} S(z)\, J_{mB}\left(mB\frac{z}{z^*}\right) \mathrm{d}z, \qquad \text{(A 1)}$$

where z^* is supposed less than unity, but greater than z_0, and $S(z)$ is a sufficiently slowly varying function.

Following Hawkings & Lowson (1974), we use the integral representation of the Bessel function (Abramowitz & Stegun 1968)

$$J_{mB}\left(mB\frac{z}{z^*}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(\mathrm{i}mB\left[\frac{z}{z^*}\sin\phi - \phi\right]\right) \mathrm{d}\phi, \qquad \text{(A 2)}$$

so that I is written as a double integral

$$I = \frac{1}{2\pi}\int_{z_0}^{1}\int_{-\pi}^{\pi} \frac{S(z)}{z} \exp\left(\mathrm{i}mB\left[\frac{z}{z^*}\sin\phi - \phi\right]\right) z\, \mathrm{d}\phi\, \mathrm{d}z, \qquad \text{(A 3)}$$

where the ϕ integral can be interpreted as an integration along the propeller advance path. Application of the method of stationary phase (see Jones 1966), noting that the integrand has a stationary phase point at $z = z^*$, $\phi = 0$, yields that

$$I \sim S(z^*)\frac{z^*}{|mB|} + \dots . \qquad \text{(A 4)}$$

Crighton & Parry (1990*b*) have calculated the next term in the series, and have proved it to be $O(mB)^{-\frac{3}{2}}$ and to be associated with the blade tip $z = 1$.

Appendix B

The condition $k_y+k_x X_Y=0$, where all quantities are evaluated at the Mach radius, reduces to

$$\cos\theta - M_x + X_Y \sin\theta = 0, \tag{B 1}$$

and without loss of generality we consider the leading-edge case. The velocity component towards the observer of the shock at the airfoil leading edge is then

$$v_{\mathrm{obs}} = V\{\cos\theta\cos(\alpha+\delta)+\sin\theta\sin(\alpha+\delta)\}, \tag{B 2}$$

where V is the normal velocity of the shock at the leading edge, and δ is given by

$$\tan\delta = X_Y. \tag{B 3}$$

Strictly, θ is the angle between the x-axis and the observer-hub vector, but will reduce to that between the x-axis and observer-source vector in the far field. Simple manipulation, and use of (B 1), will now yield that

$$\frac{v_{\mathrm{obs}}}{c_0} = \frac{V}{c_0 M_{\mathrm{r}}\cos\delta}, \tag{B 4}$$

and the condition that the shock remains attached, i.e.

$$V = c_0 M_{\mathrm{r}} \cos\delta, \tag{B 5}$$

demonstrates that $v_{\mathrm{obs}} = c_0$, and thus the velocity component (in the observer direction) of the shock at the blade leading edge is exactly sonic.

Appendix C

The radiation integral for the mth harmonic of steady loading noise can be shown to be

$$P_{\mathrm{sl}}^{m} = \frac{B\exp\left(\mathrm{i}mB\left[\tfrac{1}{2}\pi-\dfrac{M_{\mathrm{t}}\tilde{r}_0}{1-M_x\cos\theta}\right]\right)}{4\pi\tilde{r}_0(1-M_x\cos\theta)}\int_{z_0}^{1}\int_{-\frac{1}{2}}^{\frac{1}{2}} J_{mB}\left(\frac{mBM_{\mathrm{t}}\sin\theta}{1-M_x\cos\theta}\right)$$
$$\times[(1+|\nabla h|^2)]^{\frac{1}{2}}\,\mathrm{i}\boldsymbol{k}\cdot\{\boldsymbol{f}_{\mathrm{U}}\exp(-\mathrm{i}k_y h/c)+\boldsymbol{f}_{\mathrm{L}}\exp(\mathrm{i}k_y h/c)\}\exp(-\mathrm{i}k_x X)\,\mathrm{d}X\,\mathrm{d}z, \tag{C 1}$$

where the wave vector $\boldsymbol{k}$ has components (k_x, k_y) in the plane of blade cross-section, $2h(X)$ is the thickness of the (symmetric) airfoil and $\boldsymbol{f}_{\mathrm{U}}$ and $\boldsymbol{f}_{\mathrm{L}}$ are the forces per unit area exerted by the fluid on the upper ($Y>0$) and lower ($Y<0$) surfaces of the blade respectively. The derivation of this formula, in which no assumption has been made about the magnitude of the thickness h, will be given in Peake & Crighton (1991). Hanson's (1980) formula can be regained by taking the thin-blade limit (equivalent to the first term in a power series expansion with $k_y h/c$ small) and noting that in the limit $h\to 0$

$$\boldsymbol{k}\cdot\{\boldsymbol{f}_{\mathrm{U}}\exp(-\mathrm{i}k_y h/c)+\boldsymbol{f}_{\mathrm{L}}\exp(\mathrm{i}k_y h/c)\}\to k_y\,\Delta p, \tag{C 2}$$

where Δp is the pressure jump across the (thin) blade, which is related to the usual lift coefficient by

$$\Delta p = \tfrac{1}{2}\rho_0 U_{\mathrm{r}}^2 c_{\mathrm{L}}. \tag{C 3}$$

This expansion, however, is not valid when $k_y h/c$ is no longer small, and will certainly break down for large values of m (see (2)); a point noted by Hanson (1980). Use of the thin-blade approximation will therefore lead to errors in the higher derivatives of the steady loading noise real-time waveform, and in its arrival time (owing to the misplacement of the singularities); these errors will be of no consequence in measures of the most significant lower harmonics, and the correct form of the higher frequency tones could be recovered from (C 1) (which is of course exact), together with an estimate of the force distribution on the blades (obtained from steady aerodynamic codes).

The form of (C 1), and its dependence on the parameter $k_y h/c$, can be understood as follows. As a source passes through the Ffowcs Williams & Hawkings sonic condition it will be non-compact on the Mach wavelength scale, seemingly ruling out application of the thin-blade approximation. However, the non-zero blade rotation will in fact act to limit this Mach radiation. The (finite) pressure level is governed by the time-scale of the source acceleration through the sonic condition, as demonstrated by Ffowcs Williams & Hawkings (1969) and as is evident in the preceding analysis from the dominance of equation (A 3) by some asymptotically small (but non-zero) region around the stationary phase point $\phi = 0$. Since the whole of the source history is included in the derivations of (1) and (C 1), the above mechanism is fully accounted for, and the asymptotic formulae calculated from them genuinely represent the finite Mach radiation. From (C 1) it is clear that what actually governs the applicability of the thin-blade approximation is whether or not the blade section is compact on the *transverse* wavelength (i.e. k_y^{-1}) scale, or equivalently whether the retarded time difference between sources on opposite blade surfaces (but at the same chordwise position) can be ignored. The wavenumber k_y is Doppler shifted, but, as might now be expected, only by the steady rectilinear component of the motion (leading to the factor $(1-M_x \cos\theta)^{-1}$ in (2)), and since the flight Mach number is taken as subsonic, k_y can never be infinite. The non-uniformity in Hanson's expansion of (C 1) for small h therefore arises only for very large m and, as stated, only introduces errors in the arrival time and high derivatives of the pressure.

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